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REFINABLE MAPS AND SHAPE

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In [9], J. J. Kelley defined very important notion "property [K] and he proved that if X is a continuum which has property [K], then the hyperspace C(X) of subcontinua of X is contractible. In [16], R. W. Wardle proved that every confluent map preserves property [K]. It is well-known that every refinable map is weakly confluent (see [1]), but simple examples show that weakly confluent maps do not preserve property [K]. In [12, (16.38) Question], S. B. Nadler asked the following question; what kinds of mappings preserve property [K] ? We show that every refinable map preserves property [K]. In [1], J. Ford and J. W. Rogers proved that every refinable map onto a Peano continuum (locally connected) is monotone. In [10], S. B. Nadler proved that if f: $X \to Y$ is a near-homeomorphism between compacta and Y has property [K], then f is confluent. Note that every near-homeomorphism is a refinable map but the converse is not true. We show that if r: $X \to Y$ is a refinable map between compacta and Y has property [K], then r is confluent. The condition that Y has property [K] cannot be omitted. We give a example in which refinable maps are not confluent. Also, we show that if r: $X \to Y$ is a refinable map between continua, then X is irreducible iff Y is irreducible. Moreover, in shape theory, we have the following: If r: $X \to Y$ is a refinable map between compacta and Y is calm, then r is a shape equivalence. As a corollary, if r: $X \to Y$ is a refinable map between compacta and either X or Y is S^n-like ($n \geq 1$), then r is a shape equivalence, where S^n denotes the n-sphere (cf. [3]). Several properties concerning refinable maps have been studied in ([1,2,3,4,5,6,7,13, etc.]).

The word compactum means a compact metric space. A connected compactum is called a continuum. If x and y are points of a metric space, $d(x,y)$ denotes the distance from x to y. For any subsets A, B of a metric space, let $d(A,B) = \inf\{d(a,b) \mid a \in A, b \in B\}$. Also, let $d_H(A,B) = \max\{\sup_{a \in A} d(a,B), \sup_{b \in B} d(b,A)\}$. d_H is called the Hausdorff metric (see [9], [12]). A compactum X is said to have property [K] (see [9]) provided that given $\varepsilon > 0$ there exists $\delta > 0$ such that if $a, b \in X$, $d(a,b) < \delta$, and A is a subcontinuum of X with $a \in A$, then there exists a subcontinuum B of X such that $b \in B$ and $d_H(A,B) < \varepsilon$. Note that every locally connected compactum has a property [K], but the converse is not true. A map $f: X \to Y$ between compacta is confluent (weakly confluent) if for every subcontinuum Q of Y each (at least one, respectively) component of the inverse image $f^{-1}(Q)$ is mapped by f onto Q. A map $r: X \to Y$ between compacta is refinable [1] if for every $\varepsilon > 0$ there exists an onto map $f: X \to Y$ such that $\operatorname{diam} f^{-1}(y) < \varepsilon$ for each $y \in Y$ and $d(r,f) = \sup \{d(r(x), f(x)) \mid x \in X\} < \varepsilon$. By definitions, each refinable map is surjective, each near-homeomorphism is refinable and if there is a refinable map from a compactum X to a compactum Y, then X is Y-like (see [5] for the definition that X is Y-like). But any converse assertions of them are not true.

Theorem. Let $r: X \to Y$ be a refinable map between compacta. If X has property [K], then Y has the same property.

Corollary. If $r: X \to Y$ is a refinable map between continua and X has property [K], then the hyperspaces 2^Y and $C(Y)$ are contractible.

Theorem. Let $r: X \to Y$ be a refinable map between compacta. If Y has property [K], then r is confluent.

Remark. In the statement of above theorem, we cannot omit the condition that Y has property [K]. In the plane R^2, put

$$A = \{(2,y) \mid -1 \leqq y \leqq 2\},$$
$$B = \mathrm{Cl}\{(x, \sin[2\pi/x]) \mid -1 \leqq x < 0\},$$
$$C = \mathrm{Cl}\{(x, \sin[2\pi/x]) \mid 0 < x \leqq 1\},$$
$$D = \mathrm{Cl}\{(x, \sin[2\pi/x-2]) \mid 1 \leqq x < 2\}, \quad \text{and}$$
$$E = \{(0,y) \mid -1 \leqq y \leqq 2\}.$$

Also, let $X = A \cup B \cup C \cup D$ and $Y = B \cup E$. Define a map $r: X \to Y$ by

$$r(p) = \begin{cases} (0,y) & \text{if } p=(2,y) \in A, \\ (0, \sin[2\pi/x]) & \text{if } p=(x, \sin[2\pi/x]) \in C, \\ (0, \sin[2\pi/x-2]) & \text{if } p=(x, \sin[2\pi/x-2]) \in D, \\ p & \text{if } p \in B. \end{cases}$$

Then it is easily seen that r is a refinable map, but not confluent.

Corollary. If $r: X \to Y$ is a refinable map between compacta and X has property [K], then r is confluent.

It is well-known that the condition that the hyperspaces 2^X and $C(X)$ of a continuum X is contractible does not imply that X has property [K]. Hence, the following question is raised.

Question. Let $r: X \to Y$ be a refinable map between continua. If the hyperspaces 2^X and $C(X)$ are contractible, are the hyperspaces 2^Y and $C(Y)$ contractible ?

Recall that a continuum X is irreducible if there exist two points $p, q \in X$ such that no proper subcontinuum of X contains p and q. A continuum is hereditarily decomposable (hereditarily indecomposable) if for any non-degenerate subcontinuum A of X, there exists (there does not exist) a decomposition of A into two proper subcontinua A_1 and A_2 of A such that $A = A_1 \cup A_2$. A continuum T is a triod if there are three subcontinua A, B and C of T such that $T = A \cup B \cup C$, $A \cap B \cap C = A \cap B = B \cap C = C \cap A$ and this common part is a proper subcontinuum of each of them. A continuum is atriodic if X fails to contain a triod ([12]).

Theorem. Let $r: X \to Y$ be a refinable map between continua. Then X is irreducible iff Y is irreducible.

To prove the above theorem, we need the following characterization of irreducible continua.

Theorem (R. H. Sorgenfrey [15]). A necessary and sufficient condition that X is irreducible is that if X is the essential sum of three proper subcontinua, then some pair fails to intersect.

Proposition. Let $r: X \to Y$ be a refinable map between compacta. If either X or Y is a Cantor set, then r is a near-homeomorphism, i.e., X and Y are Cantor sets.

Proposition. Let $r: X \to Y$ be a refinable map between continua. Then

(1) if X is hereditarily decomposable, then Y is also,

(2) X is hereditarily indecomposable iff Y is also, and

(3) X is atriodic iff Y is also.

Corollary. Let $r: X \to Y$ be a refinable map between continua. If either X or Y is the pseudo-arc, then r is a near-homeomorphism, i.e., X and Y are pseudo-arcs.

A compactum X is calm if whenever $X \subset M \in$ ANR, there is a neighborhood V of X in M such that for any neighborhood U of X in M there is a neighborhood W of X in M, $W \subset U$ such that if $f, g: Y \to W$ are maps with $f \simeq g$ in V, then $f \simeq g$ in U for all $Y \in$ ANR.

Theorem. If $r: X \to Y$ is a refinable map between compacta and Y is calm, then r is a shape equivalence, i.e., $sh(X)=sh(Y)$.

Corollary. If $r: X \to Y$ is a refinable map between compacta and Y is an FANR, then r is a shape equivalence (see [3]).

Corollary. If $r: X \to Y$ is a refinable map between compacta and Y is an $AANR_N$, then r is a shape equivalence.

Remark. In the statements of above results, we cannot replace "calm" by "movable". Also, we cannot replace "$AANR_N$" by "$AANR_C$" (see [4]).

As a corollary, we have

Corollary. If r: $X \to Y$ is a refinable map between compacta and if either X or Y is S^n-like ($n \geq 1$), then r is a shape equivalence, where S^n denotes the n-sphere.

Question. Does every refinable map preserve calmness (FANR, $AANR_N$) ?

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